Discrete-Time Sliding Mode Control
for State Delay Systems using Nonlinear Sliding Surface

Machhindranath Patil\(^1\) and B. Bandyopadhyay\(^2\)
\(^1\)Indian Institute of Technology Bombay, Mumbai, India
(E-mail: machhindra@sc.iitb.ac.in)
\(^2\)Senior IEEE member, Indian Institute of Technology Bombay,
Mumbai, India
(E-mail: bijnan@ee.iitb.ac.in)

Abstract: This paper considers the nonlinear surface design for uncertain discrete time system with state delay. Using Lyapunov-Krasovskii method delay-independent conditions are derived for the stable sliding motion along the nonlinear sliding surface. The sliding surface is typically designed to obtain the high speed response without exhibiting the overshoot. This can be achieved by keeping the damping ratio initially small and as the trajectory approaches to the origin, damping ratio is made high to avoid the overshoot in response.

Keywords: Sliding mode control, Lyapunov-Krasovskii method, Composite nonlinear feedback, Delay systems.

1. INTRODUCTION

Time delay is commonly encountered in various electrical, hydraulic, mechanical systems and very often in chemical processes due to measurement, transmission and transportation lags, unmodelled inertia or computational delay. The existence of the delay often degrades the performance of the system and sometimes it leads to the instability. Input delay is caused by the transmission of a control signal over a long distance, state delay is a result of transmission or transport delay among interacting elements in a dynamic system and the output delay is the delay resulting from sensors.

The stabilization of time delay systems can be classified into two types: delay independent stabilization and delay-dependent stabilization. In the delay-independent stabilization a controller is designed which can stabilize a system regardless of the amount delay. On the other hand, the delay dependent stabilization is achieved with a controller which considers the size of the delay.

In recent times sliding mode control (SMC) for the time delay systems has been topic of great interest for the researchers as it offers high performance and robustness. An approach based on lumped function is adopted in [6] to design a SMC law for systems with state delay.\(^7\) have proposed the control law for systems where delay is present in both input and state. A predictor based approach for discrete time SMC for input delay systems is proposed in [8]. In [20] the discrete SMC design based on multirate output feedback (MROF) for delay systems considering four different structures is proposed. In all above cited papers the sliding mode control uses the linear sliding surface.

Bandyopadhyay and Fulwani proposed the nonlinear sliding surface design for discrete time system with input delay in which predictor is used to convert the system into delay-less system, see [22]. The nonlinear sliding surface design is based on the composite nonlinear feedback (CNF) technique.

To improve the step response of second order linear systems with actuator saturation in terms of settling time and overshoot specifications, an idea of using CNF controller is proposed in [15] and further these results are extended to the higher order and multiple input systems by [16]. CNF controller consists of a linear feedback law and a nonlinear feedback law without any switching element. The linear feedback part is designed to yield a closed-loop system with a small damping ratio for a quick response. The nonlinear feedback law is used to increase the damping ratio of the closed-loop system to reduce the overshoot caused by the linear part as the system output approaches the target reference , see also [17], [18].

In this paper the delay-independent conditions for the stability of the subsystem in sliding motion using Lyapunov-Krasovskii method are derived.

Abbreviations and Acronyms:
\(-\text{Real vector space, } I - \text{Identity matrix, } M^T - \text{Transpose of } M, M \cdot N - \text{post-multiplying } N \text{ to } M, \lambda_{\text{max}}(M) - \text{largest eigen value of matrix } M, \lambda_{\text{min}}(M) - \text{smallest eigen value of matrix } M, \sigma_{\text{max}}(M) - \text{largest singular value of matrix } M, \sigma_{\text{min}}(M) - \text{smallest singular value of matrix } M \text{ and } \|M\| - \text{Euclidean norm of } M.\)

2. PROBLEM FORMULATION

Consider the following uncertain discrete-time linear system with state delay.

\[ x(k+1) = \Phi x(k) + \Theta x(k-h) + \Gamma u(k) + D p(k) \]  
\[ y(k) = C_1 x(k) \]  
\[ x(k) = \Theta(k) \cdot k = -h, -h+1, \cdots 0 \]  

Where, \( x(k) \in \mathbb{R}^n, u(k) \in \mathbb{R}, y(k) \in \mathbb{R} \) are state, input and output of the system respectively. \( \Theta(k) \) is an initial condition and \( h \) is amount of delay.

Bringing the system into regular form by the transformation \( z(k) = T_r x(k) \) such that \( T_r \Gamma = \begin{bmatrix} 0 & \Gamma_2 \end{bmatrix}^T \).
$T_r D p(k) = \left[ \begin{array}{cc} p_1(k) & p_2(k) \end{array} \right]^T$, $\Phi_{reg} = T_r \Phi T_r^{-1}$ and $\Phi_{reg} = T_r \Phi T_r^{-1}$.

Define, $z := [z_1(k) z_2(k)]^T$, $C := C_1(T_r)^{-1}$

$\Phi_{reg} := \left[ \begin{array}{cc} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{array} \right]$, $\Phi_{reg} := \left[ \begin{array}{cc} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{array} \right]$

Then the system in regular form is represented as,

$z_1(k+1) = \Phi_{11} z_1(k) + \Phi_{12} z_2(k) + \Phi_{11} z_1(k-h) + p_1(k)$
$z_2(k+1) = \Phi_{21} z_1(k) + \Phi_{22} z_2(k) + \Phi_{21} z_1(k-h) + \Phi_{22} z_2(k-h) + \Gamma_2 u(k) + \rho_2(k)$

Equation (4)

$y(k) = C z(k)$

(6)

Where $p_1(k)$ and $\rho_2(k)$ are unmatched and matched uncertainty components of the disturbance term, respectively. Associated with the given system following assumptions are to be made.

Assumptions 1: $(\Phi, \Gamma')$ pair is stabilizable and $(\Phi, C_1)$ pair is observable.

Assumptions 2: There exists the matrix $F$ such that $(\Phi_{11} - \Phi_{12} F)$ and $(\Phi_{11} - \Phi_{12} F)$ have stable dynamics.

### 3. NONLINEAR SLIDING SURFACE

For the tracking problem the desired trajectory needs to be generated consistently. As $(\Phi, \Gamma')$ pair is assumed to be stabilizable, there exists the control $u_d(k)$ that generates the desired trajectory $z_d := [z_1d(k)^T z_2d(k)^T]^T$ such that

$z_{1d}(k+1) = \Phi_{11} z_{1d}(k) + \Phi_{12} z_{2d}(k)$
$z_{2d}(k+1) = \Phi_{21} z_{1d}(k) + \Phi_{22} z_{2d}(k)$

Equation (7)

Equation (8)

Define,

$e(k) := \left[ \begin{array}{cc} e_1(k) \\ e_2(k) \end{array} \right] := \left[ \begin{array}{cc} z_1(k) - z_{1d}(k) \\ z_2(k) - z_{2d}(k) \end{array} \right]$

Equation (9)

Then from eqs. (4) - (9) we obtain the tracking system

$e_1(k+1) = \Phi_{11} e_1(k) + \Phi_{12} e_2(k) + \Phi_{11} z_1(k-h)$
$e_2(k+1) = \Phi_{21} e_1(k) + \Phi_{22} e_2(k) + \Phi_{21} z_1(k-h)$

Equation (10)

Equation (11)

Define,

$c_1(k) := F - \Psi(y(k)) \Phi_{12} P(\Phi_{11} - \Phi_{12} F)$
$c^T(k) := \left[ \begin{array}{cc} c_1(k) & 1 \end{array} \right]$

Equation (12)

Equation (13)

Where $P$ is some positive definite symmetric matrix (see, Remark 1) and $\Psi(y(k))$ is non-positive nonlinear function which is used to change the damping ratio of reduced order subsystem as the output approaches the reference input. The nonlinear function is chosen such that its value changes from 0 to $-\beta$ that changes the initial gain $c_1(k) = F$ designed for small risetime to the $c_1(k) = K$ to yield overdamped response as tracking error tends to zero.

Possible choice of $\Psi(y(k))$ is as follows,

$\Psi(y(k)) = -\beta \exp(-\bar{k} |y(k-1) - r(k)|)$

Equation (14)

Where $\beta$ and $\bar{k}$ are tuning parameters. The value of $\beta$ contributes the change in controller gain whereas $\bar{k}$ determines the rate of change of $\Psi(y(k))$.

Thus the nonlinear sliding surface for the system in regular form (27)-(11) is given by

$s(k) := c^T(k) e(k)$

Equation (15)

$s(k) = c_1(k) z_1(k) + z_2(k) - c^T z_d(k)$

Equation (16)

Remark 1: $P = P^T > 0$ can be obtained from the solution of lyapunov equation or riccati equation along the solution of the closed loop systems satisfying the condition $K = F + \beta \Phi_{12} P(\Phi_{11} - \Phi_{12} F)$. Such a $P$ exists because of the assumption-2. However, it is not always feasible to compute $P$ and $\beta$ satisfying this condition and Lyapunov equation. The choice of $P$ and $\beta$ is discussed in section 5.3.

### 4. CONTROL LAW

The control law for the system in the regular form is derived using reaching law [25]. To reach the sliding surface in one sampling period the reaching law is given by

$s(k+1) = 0$

Equation (17)

From eq. (16),

$s(k+1) = c_1(k+1) \Phi_{11} z_1(k)$
$+ c_1(k+1) \Phi_{12} z_2(k)$
$+ c_1(k+1) \Phi_{11} z_1(k-h)$
$+ c_1(k+1) \Phi_{12} z_2(k-h)$
$+ c_1(k+1) \rho_1(k) + \Phi_{21} z_1(k) + \Phi_{22} z_2(k)$
$+ \Phi_{21} z_1(k-h) + \Phi_{22} z_2(k-h) + \Gamma_2 u(k)$
$+ \rho_2(k) - c^T(k+1) z_d(k+1)$

Equation (18)

The control law obtained from (17) and (18) is given by,

$u(k) = -(\Gamma_2)^{-1} \left[ c_1(k+1) \Phi_{11} z_1(k)$
$+ c_1(k+1) \Phi_{12} z_2(k)$
$+ c_1(k+1) \Phi_{11} z_1(k-h)$
$+ c_1(k+1) \Phi_{12} z_2(k-h)$
$+ \Phi_{21} z_1(k) + \Phi_{22} z_2(k) + \Phi_{21} z_1(k-h)$
$+ \Phi_{22} z_2(k-h) + c_1(k+1) \rho_1(k-1)$
$+ \rho_2(k-1) - c^T(k+1) z_d(k+1) \right]$

Equation (19)

The control law (19) tends the trajectories towards the sliding surface and forces the trajectory to remain in quasi sliding mode band.

Remark 2: The control law as stated in (19) considers estimation of uncertainty at just previous instant $p_1(k-1)$ and $\rho_2(k-1)$ as proposed in [5]. The uncertain terms
\( \rho_1(k) \) and \( \rho_2(k) \) are estimated from their values at just previous instant.

Computing the uncertainty from the eq. (4) and (5),

\[
\rho_1(k-1) = z_1(k) - \Phi_{11} z_1(k-1) - \Phi_{12} z_2(k-1) \\
- \Phi_{111} z_1(k - h - 1) \\
- \Phi_{122} z_2(k - h - 1) \tag{20}
\]
\[
\rho_2(k-1) = z_2(k) - \Phi_{211} z_1(k-1) - \Phi_{222} z_2(k-1) \\
- \Phi_{211} z_1(k - h - 1) - \Phi_{222} z_2(k - h - 1) \tag{21}
\]

5. STABILITY OF THE SLIDING MOTION

5.1 Boundedness of sliding surface

The sliding mode control guarantees robustness of matched uncertainty with known bounds. There is no such guarantee for unmatched uncertainty. It is evident that for the discrete-time uncertain systems the sliding motion is only possible in small boundary layer around the sliding surface. By the application of the control law (19) in eq. (18), the sliding surface dynamics is appeared as,

\[
s(k + 1) = c_1(k + 1)(\rho_1(k) - \rho_1(k - 1)) + \rho_2(k) - \rho_2(k - 1) \tag{22}
\]
Assume that \( \|c_1(k+1)\| \) is bounded by the positive scalar \( \epsilon_{cm} \). For bounded rate of change of disturbance there exists the positive scalars \( d_1 \) and \( d_2 \) such that following inequalities are satisfied.

\[
|\epsilon_{cm}(\rho_1(k) - \rho_1(k-1))| \leq d_1 \quad \forall k \\
|\rho_2(k) - \rho_2(k - 1)| \leq d_2 \quad \forall k \tag{23}
\]

It implies that the sliding surface \( s(k) \) satisfies the bound \( |s(k)| \leq d_1 + d_2 \) \tag{25}

5.2 Boundedness of reduced order system during sliding mode

To verify the stability of the sliding motion we fist obtain the closed loop dynamics of the subsystem.

From (16),

\[
e_2(k) = -e_1(k)e_1(k) + s(k) \tag{26}
\]
So the subsystem dynamics during sliding motion is given by

\[
e_1(k + 1) = \Phi_{11} e_1(k) + \Phi_{12} s(k) + \Phi_{111} e_1(k - h) \\
- \Phi_{12}(F - \Psi(y(k))\Psi_{12} P(\Phi_{11} - \Phi_{12} F)) e_1(k) + (\Phi_{11} - \Phi_{12} F)e_1(k - h) \\
+ \Phi_{111} e_1(k - h) + \rho_1(k) \tag{27}
\]

Define,

\[
\Phi_{11eq} := (\Phi_{11} - \Phi_{12} F) \tag{28}
\]
\[
\Phi_{12eq} := (\Phi_{12} - \Phi_{11} F) \tag{29}
\]
\[
F_n := \Phi_{12}(y(k))\Psi_{12} P\Phi_{11eq} \tag{30}
\]
\[
F_n := \Phi_{12}(y(k))\Psi_{12} P\Phi_{11eq} \tag{31}
\]
\[
g(k) := \Phi_{12}(s(k) + \Phi_{111} e_1(k - h) + \rho_1(k) \tag{32}
\]

So from equation (27) reduced order system during the sliding motion is given by,

\[
e_1(k + 1) = \Phi_{11eq} e_1(k) + F_n e_1(k) + \Phi_{11eq} e_1(k - h) + F_n e_1(k - h) + g(k) \tag{33}
\]

The Lyapunov-Krasovskii functional \( V(x(k)) \) for the subsystem (33) is given by

\[
V(k) = e_1^T(k) P e_1(k) + \sum_{j=1}^h e_1^T(k - j) W e_1(k - j) \tag{34}
\]

The forward difference along the solutions of system is

\[
\Delta V(k) = e_1^T(k + 1) P e_1(k + 1) - e_1^T(k) P e_1(k) + e_1^T(k) W e_1(k) - e_1^T(k - h) W e_1(k - h) \tag{35}
\]

\[
\Rightarrow \Delta V(k) = \{\Phi_{11eq} e_1(k) + F_n e_1(k) + \Phi_{11eq} e_1(k - h) + F_n e_1(k - h) + g(k)\}^T P \{\Phi_{11eq} e_1(k) + F_n e_1(k) + \Phi_{11eq} e_1(k - h) + F_n e_1(k - h) + g(k)\} \tag{36}
\]

Define,

\[
v_1(k) := [e_1^T(k) \quad e_1^T(k - h)]^T \tag{37}
\]
\[
v_2(k) := \{F_n e_1(k) + \Phi_{11eq} e_1(k - h)\}^T P \{F_n e_1(k) + \Phi_{11eq} e_1(k - h)\} \tag{38}
\]
\[
v_3(k) := \{\Phi_{11eq} e_1(k) + \Phi_{11eq} e_1(k - h)\}^T P \{\Phi_{11eq} e_1(k) + \Phi_{11eq} e_1(k - h)\} \tag{39}
\]
\[
v_4(k) := \{F_n e_1(k) + \Phi_{11eq} e_1(k - h)\}^T P \{F_n e_1(k) + \Phi_{11eq} e_1(k - h)\} \tag{40}
\]
\[
g(k) := \{F_n e_1(k) + \Phi_{11eq} e_1(k - h)\}^T P g(k) \tag{41}
\]

Substituting definitions (37) - (41) into eq. (36)

\[
\Delta V(k) = v_1(k) + v_2(k) + v_3(k) + v_4(k) + g(k) \tag{42}
\]

Using the fact as given in [14] that for any Positive definite symmetric matrix \( P \) and any matrices \( F \) and \( G \) of appropriate dimensions following inequality holds,

\[
(F + G)^T P (F + G) \leq (1 + \varepsilon) F^T PF + (1 + \varepsilon^{-1}) G^T PG \tag{43}
\]
Where $\varepsilon$ is a small positive constant.

Using an identity in (43), $v_2(k)$ from eq. (38) can be written as

$$v_2(k) \leq \left(1 + e^2\right) e_1^T(k) F_n^T P F_n e_1(k) + \left(1 + e^{-1}\right) e_1^T(k - h) F_n^T P F_n e_1(k - h)$$

(44)

Define,

$$L := \left(1 + e^2\right) F_n^T P F_n$$

$$= \left(1 + e^2\right) \Phi_{11eq}^T \Phi_{12} \psi(g(k)) \Phi_{12}^T$$

$$\leq \beta^2 \left(1 + e^2\right) \Phi_{11eq}^T \Phi_{12} \Phi_{12}^T \Phi_{11eq} \Phi_{11eq}$$

(45)

$$\bar{L} := \left(1 + e^{-1}\right) F_n^T P F_n$$

$$= \left(1 + e^{-1}\right) \Phi_{11eq}^T \Phi_{12} \psi(g(k - h)) \Phi_{12}^T$$

$$\leq \beta^2 \left(1 + e^{-1}\right) \Phi_{11eq}^T \Phi_{12} \Phi_{12}^T \Phi_{11eq} \Phi_{11eq}$$

(46)

Substituting in (44),

$$v_2(k) \leq \left\{ \left[ e_1^T(k) \quad e_1^T(k - h) \right] \right\} \left[ \begin{array}{cc} L & 0 \\ 0 & \bar{L} \end{array} \right]$$

$$\leq \left\{ \left[ e_1^T(k) \quad e_1^T(k - h) \right] \right\}$$

(47)

**Theorem 1**

For some $P = P^T > 0$ and $W = W^T > 0$,

$$v_1(k) + v_2(k) < 0$$

(48)

if following linear matrix inequality (LMI) holds

$$\begin{bmatrix} -P + W + L & 0 \\ * & -W + \bar{L} \end{bmatrix} \begin{bmatrix} \Phi_{11eq}^T P \\ \Phi_{11eq} \end{bmatrix} < 0$$

(49)

Where $v_1(k), v_2(k), L$ and $\bar{L}$ are as defined in (37), (47), (45) and (46).

**Proof**

$$v_1(k) + v_2(k) \leq \left\{ \left[ e_1^T(k) \quad e_1^T(k - h) \right] \right\}$$

$$\leq \left\{ \left[ e_1^T(k) \quad e_1^T(k - h) \right] \right\}$$

Where

$$D_1 = \Phi_{11eq}^T \Phi_{12} \psi(g(k)) \Phi_{12}^T$$

$$D_2 = \Phi_{11eq}^T \Phi_{11eq}$$

$$D_3 = \Phi_{11eq}^T \Phi_{11eq}$$

$$D_4 = \Phi_{11eq}^T \Phi_{11eq}$$

Define,

$$\bar{P} := \begin{bmatrix} D_1 & D_2 \\ D_3 & D_4 \end{bmatrix}$$

$$= \begin{bmatrix} -P + W + L & 0 \\ * & -W + \bar{L} \end{bmatrix}$$

$$\leq \begin{bmatrix} \Phi_{11eq}^T P \\ \Phi_{11eq} \end{bmatrix}$$

$$\leq \begin{bmatrix} \Phi_{11eq}^T P \\ \Phi_{11eq} \end{bmatrix}$$

(50)

Pre and post Multiplication of $\text{diag}\{I, I, P\}$ to the condition (50) gives the LMI

$$\begin{bmatrix} -P + W + L & 0 \\ * & -W + \bar{L} \end{bmatrix} \left[ \begin{array}{cc} \Phi_{11eq}^T P \\ \Phi_{11eq} \end{array} \right] < 0$$

(51)

If the LMI condition in (51) is satisfied then,

$$v_1(k) + v_2(k) < -\lambda_{\text{min}}(\bar{P}) \|e_1(k)\|^2$$

(52)

This completes the proof.

In order to prove the boundedness of $e(k)$, the following lemmas are necessary.

**Lemma 2**

For a vector $g(k)$ as defined in (32),

$$\|g(k)\| \leq g_m$$

(53)

where $g_m = (d_1 + d_2) \|\Phi_{12} + \Phi_{11eq}\| + \rho_{1m}$ and $\rho_{1m} > 0$ is unmatched uncertainty bound.

**Proof**

By definition,

$$g(k) = \Phi_{11eq} \tilde{s}(k) + \Phi_{11eq} s(k - h) + \rho_1(k)$$

$$\Rightarrow \|g(k)\| \leq \|\Phi_{11eq} \tilde{s}(k)\| + \|\Phi_{11eq} s(k - h)\| + \|\rho_1(k)\|$$

As uncertainty is bounded, there exists a positive scalar $\rho_{1m}$ such that $\|\rho_1(k)\| \leq \rho_{1m}$. From (25), $|s(k)| < (d_1 + d_2)$. Hence

$$g(k) \leq (d_1 + d_2) \|\Phi_{12} + \Phi_{11eq}\| + \rho_{1m}$$

$$\Rightarrow g(k) \leq g_m$$

**Lemma 3**

Vectors $\alpha(k) := g^T(k) P (F_n + \Phi_{11eq})$ and $\bar{\alpha}(k) := g^T(k) P (F_n + \Phi_{11eq})$ are bounded by

$$\|\alpha(k)\| \leq g_m \lambda_{\text{max}}(P) \alpha_m$$

(54)

$$\|\bar{\alpha}(k)\| \leq g_m \lambda_{\text{max}}(P) \bar{\alpha}_m$$

(55)

Where

$$\alpha_m = \|\beta \Phi_{12} \Phi_{11eq} P \Phi_{11eq}\| + \|\Phi_{11eq}\|$$

$$\bar{\alpha}_m = \|\beta \Phi_{12} \Phi_{11eq} P \Phi_{11eq}\| + \|\Phi_{11eq}\|, F_n, F_n$$

are as defined in eqs. (29) - (31).

**Proof**

By definition $|\psi(k)| < \beta$ and from lemma 5, $\|g(k)\| \leq g_m$. It follows,

$$\alpha(k) = g^T(k) P (F_n + \Phi_{11eq})$$

$$= g^T(k) P (\Phi_{12} \psi(k) \Phi_{12}^T P \Phi_{11eq} + \Phi_{11eq})$$

$$\Rightarrow \|\alpha(k)\| \leq \|g^T(k) P\| \|\Phi_{12} \psi(k) \Phi_{11eq}\| + \|\Phi_{11eq}\|$$

$$\Rightarrow \|\alpha(k)\| \leq \|g^T(k) P\| \|\beta \Phi_{12} \Phi_{11eq} P \Phi_{11eq}\| + \|\Phi_{11eq}\|$$

Hence $\|\alpha(k)\| \leq g_m \lambda_{\text{max}}(P) \alpha_m$. 

Similarly
\[ \alpha'(k) = g^T(k)Pn(F_n + \Phi_{11eq}) \]
\[ g^T(k)PF_n e_1(k) + \Phi_{11eq} g(k) \]
\[ \|\alpha'(k)\| \leq ||g^T(k)P|| \|F_n + \Phi_{11eq}\| \]
\[ + \|\Phi_{11eq}\| \]
\[ \|\alpha'(k)\| \leq ||g^T(k)P|| \|\Phi_{11eq} g(k) + \Phi_{11eq} g(k)\| ||\Phi_{11eq}\| \]
Hence \[ \|\alpha'(k)\| \leq g_m \lambda_{\text{max}}(P) \alpha_m. \]

Lemma 4
For \( g(k) \) as defined in (41), following inequality holds
\[ g(k) \leq 2g_m \lambda_{\text{max}}(P)(\alpha_m + \alpha_m) ||e_1(k)|| + \lambda_{\text{max}}(P) g_m' \] (56)

Where, \( \alpha_m \) and \( \alpha_m' \) are as defined in lemma 5.2

Proof
From (41),
\[ g(k) = \{F_n e_1(k) + \Phi_{11eq} e_1(k)\} + g(k)PF_n e_1(k) \]
\[ + \Phi_{11eq} g(k) \] (62)
\[ \Rightarrow g(k) = e_1(k) + g(k)PF_n e_1(k) \]
\[ + e_1(k) g(k)PF_n e_1(k) \]
\[ + e_1(k) + e_1(k) g(k)PF_n e_1(k) \]
\[ + e_1(k) g(k)PF_n e_1(k) \]
\[ + e_1(k) g(k)PF_n e_1(k) \]
\[ + e_1(k) g(k)PF_n e_1(k) \]
\[ \Rightarrow g(k) \leq 2g_m \lambda_{\text{max}}(P)(\alpha_m + \alpha_m) ||e_1(k)|| + \lambda_{\text{max}}(P) g_m' \]

Hence
\[ g(k) \leq 2g_m \lambda_{\text{max}}(P)(\alpha_m + \alpha_m) ||e_1(k)|| + \lambda_{\text{max}}(P) g_m' \]

Lemma 5
For \( v_3(k) \) as defined in (39), \( v_3(k) < 0 \) if
\[ \sigma_{\text{max}}(m_2 + m_4^T) < \sqrt{\lambda_{\text{min}}(m_2 + m_4^T) \lambda_{\text{min}}(m_4 + m_4^T)} \] (57)

Where,
\[ m_1 := \Phi_{11eq} P \]
\[ m_2 := \Phi_{11eq} P \]
\[ m_3 := \Phi_{11eq} P \]
\[ f_n := P \]
\[ f_n := \Phi_{11eq} P \]
\[ \delta_1 := \lambda_{\text{min}}(\tilde{P}) + 2 \lambda_{\text{max}}(M) \]
\[ \delta_2 := \lambda_{\text{max}}(P)(\alpha_m + \alpha_m) \] (65)
\textbf{Proof}

As defined in (42) the forward difference along the solutions of system (33) is
\begin{equation}
\Delta V(k) = v_1(k) + v_2(k) + v_3(k) + v_4(k) + \bar{g}(k)
\end{equation}

If condition (62) holds then from theorem 1,
\begin{equation}
v_1(k) + v_2(k) < -\lambda_{\text{min}}(\tilde{P})||e_1(k)||^2
\end{equation}

As \(v_4(k) = \tilde{v}(k)\), from lemma 5 we can write
\begin{equation}
v_3(k) + v_4(k) \leq -2\lambda_{\text{max}}(M)||e_1(k)||^2
\end{equation}

Also from lemma 4,
\begin{equation}
\bar{g}(k) \leq 2g_m\lambda_{\text{max}}(P)(\alpha_m + \bar{\alpha}_m)||e_1(k)||
+ \lambda_{\text{max}}(P)g_m^2
\end{equation}

Using conditions (67) - (69) and eq. (66), it follows that
\begin{equation}
\Delta V(k) \leq -\lambda_{\text{min}}(\tilde{P})||e_1(k)||^2 - 2\lambda_{\text{max}}(M)
\cdot ||e_1(k)||^2 + 2g_m\lambda_{\text{max}}(P)(\alpha_m + \bar{\alpha}_m)
\cdot ||e_1(k)|| + \lambda_{\text{max}}(P)g_m^2
\end{equation}

\Rightarrow \Delta V(k) \leq -\lambda_{\text{min}}(\tilde{P})||e_1(k)||^2 + 2g_m\lambda_{\text{max}}(P)(\alpha_m + \bar{\alpha}_m)||e_1(k)||
+ \lambda_{\text{max}}(P)g_m^2
\end{equation}

Substituting the stated definitions of \(\delta_1\) and \(\delta_2\),
\begin{equation}
\Delta V(k) \leq -\delta_1||e_1(k)||^2 + 2g_m\delta_2||e_1(k)||
+ \lambda_{\text{max}}(P)g_m^2
\end{equation}

Therefore \(\Delta V < 0\) for
\begin{equation}
||e_1(k)|| \geq \frac{g_m}{\delta_1} \left( \delta_2 + \sqrt{\delta_2^2 + \delta_1\lambda_{\text{max}}(P)} \right) = r
\end{equation}

Hence the trajectory originated from outside of the ball \(\Omega(0, r)\) will enter the ball asymptotically.

\section{5.3 Choice of \(P\) and \(\beta\)}

To achieve the high speed response without experiencing the overshoot, the nonlinear non-positive function \(\Psi(k)\) as defined in (14) plays vital role. As \(|\Psi(k)| \to \beta\), the feedback gain \(c_1(k) = F\) which is designed for small risetime is gradually increased till it becomes \(c_1(k) = K\) which is chosen for overdamped response. Rewriting the definition of \(c_1(k)\) from (12),
\begin{equation}
c_1(k) = F - \Psi(\bar{y}(k))\Phi_{12}^TP(\Phi_{11} - \Phi_{12}F)
\end{equation}

It implies that,
\begin{equation}
K = F + \beta\Phi_{12}^TP(\Phi_{11} - \Phi_{12}F)
\end{equation}

From (74) we can compute the value of the \(\beta\). Sometimes it is not feasible to compute \(\beta\) from eq. (74) that yields the desired damping ratio along with \(P\) that is satisfying the stability conditions (57) and (62). So the positive definite matrix \(P\) and scalar \(\beta\) are searched such that it must satisfy the stability conditions and simultaneously yield the final gain \(K_2\) close enough to the preselected value of the final gain \(K\). This can be done by formulating eq. (74) into LMI for some small positive constant \(\mu\) as follows.
\begin{equation}
H := \Phi_{12}^TP - \beta^{-1}(K - F)(\Phi_{11} - \Phi_{12}F)^{-1}
\end{equation}

\begin{equation}
\begin{bmatrix}
\mu I & H \\
H & \mu I
\end{bmatrix} > 0
\end{equation}

Solving the LMI conditions (62) and (76) along with the condition (57) for \(P\) and \(\beta\) yields the expected results.

\section{6. ILLUSTRATIVE APPLICATION}

As an illustration liquid propellant rocket motor model from [9], which they have taken from [13], is considered. In the model \(A(1, 1) = \kappa - 1\) and \(A_d(1, 1) = -\kappa\) the parameter \(\kappa\) is assumed to be 0.8. The output to be tracked is taken from the second state. The state delay is assumed to be constant and is considered as 0.4 sec. The continuous time model of the system is represented as

\begin{equation}
A = \begin{bmatrix}
-0.2 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
-1 & 0 & -1 & 1 \\
0 & 1 & 1 & 0
\end{bmatrix}
\end{equation}

\begin{equation}
A_d = \begin{bmatrix}
-0.8 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\end{equation}

\begin{equation}
B = \begin{bmatrix}
0 & 1 & 0 & 0
\end{bmatrix}^T, C_1 = \begin{bmatrix}
0 & 1 & 0 & 0
\end{bmatrix}
\end{equation}

The initial and the desired states are \(\begin{bmatrix}
0 & 0 & 0 & 0
\end{bmatrix}^T\) and \(\begin{bmatrix}
-1 & 1 & -1 & -2
\end{bmatrix}^T\) respectively.

The discrete-time version of the system with sampling period of 0.1 sec. is obtained as

\begin{equation}
\Phi = \begin{bmatrix}
0.9802 & 0 & 0 & 0 \\
0.0002 & 0.9950 & -0.0048 & -0.1000 \\
-0.0944 & 0.0048 & 0.9095 & 0.0952 \\
-0.0048 & 0.1000 & 0.0952 & 0.9998
\end{bmatrix}
\end{equation}

\begin{equation}
\bar{\Phi} = \begin{bmatrix}
-0.0792 & 0 & 0.0990 & 0 \\
0 & 0 & 0 & 0 \\
0.0038 & 0 & -0.0048 & 0 \\
0.0001 & 0 & -0.0002 & 0
\end{bmatrix}
\end{equation}

\begin{equation}
\Gamma = \begin{bmatrix}
0 & 0.0998 & 0.0002 & 0.0050
\end{bmatrix}^T
\end{equation}

To design the sliding surface the discrete-time model is transformed into the regular form

\begin{equation}
\Phi_{\text{reg}} = \begin{bmatrix}
0.9799 & -0.0046 & -0.0012 & 0.0050 \\
0.0896 & 0.9097 & 0.0996 & -0.0095 \\
0.0037 & 0.0950 & 1.0000 & -0.1003 \\
-0.0052 & 0.0002 & 0.0995 & 0.9950
\end{bmatrix}
\end{equation}

\begin{equation}
\bar{\Phi}_{\text{reg}} = \begin{bmatrix}
-0.0788 & -0.0990 & -0.0039 & 0.0002 \\
-0.0040 & -0.0050 & -0.0002 & 0 \\
-0.0041 & -0.0051 & -0.0002 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\end{equation}
\[ C = \begin{bmatrix} 0.0025 & -0.0016 & -0.0500 & -0.9987 \end{bmatrix} \]

\[ \Gamma_2 = -0.1 \]

The system is not controllable, however, uncontrollable mode at 0.9802 is stable. So the stable surface is designed via placement of the other poles. Initially poles are placed at 0.9839 + 0.0492i and 0.9839 − 0.0492i with the state feedback gain of \( F = \begin{bmatrix} 1.0748 & 2.0392 & -0.7210 \end{bmatrix} \). For the final overdamped response selecting the gain \( K = \begin{bmatrix} -0.9728 & 0.0940 & 0.9451 \end{bmatrix} \). Solving the conditions (62), (76) and (57) with \( \varepsilon = 0.01 \) and \( \mu = 0.1599 \) gives,

\[ P = \begin{bmatrix} 82.2696 & 57.8638 & -26.7928 \\ 57.8638 & 81.7835 & -28.5218 \\ -26.7928 & -28.5218 & 23.0690 \end{bmatrix} \]

and \( \beta = 0.08136 \).

The rate of change of gain is tuned with \( \bar{k} = 0.9 \). Following table shows the comparison of the performance with linear and nonlinear sliding surfaces.

<table>
<thead>
<tr>
<th>Specifications</th>
<th>Linear underdamped surface</th>
<th>Linear overdamped surface</th>
<th>Nonlinear surface</th>
</tr>
</thead>
<tbody>
<tr>
<td>Feedback gain</td>
<td>( F )</td>
<td>( K_2 )</td>
<td>Varying ( K_2 )</td>
</tr>
<tr>
<td>( \xi )</td>
<td>0.287</td>
<td>1.0</td>
<td>Varying</td>
</tr>
<tr>
<td>( t_r ) sec.</td>
<td>1.002</td>
<td>13.7495</td>
<td>9.3510</td>
</tr>
<tr>
<td>( M_p % )</td>
<td>14.1788</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( t_s ) sec.</td>
<td>14.2691</td>
<td>23.7530</td>
<td>17.8893</td>
</tr>
</tbody>
</table>

Fig.1 indicates the comparison of output \( y(k) \) of the system without disturbance with two linear sliding surfaces designed for \( \xi = 0.287 \) and \( \xi = 1 \) and a nonlinear sliding surface designed for variable damping ratio. Fig.2 and Fig.3 indicate the switching function \( s(k) \) and control input \( u(k) \) for the system without disturbance.

Consider that the disturbance \( \rho(k) = sin(0.5k) \) with \( D = \begin{bmatrix} 0 & 0.01 & 0 & 0.01 \end{bmatrix} \) is introduced. Then the sliding motion is guaranteed within the sliding mode band of \( s(k) < (d_1 + d_2) = 3.0088 \). Various bounds as
given in \textit{lemma-2} to \textit{lemma-5} are: $\gamma_m = 0.3081, \alpha_m = 1.3686, \alpha_m = 0.1267, \delta_1 = 317.6293, \delta_2 = 226.9552$. As given in \textit{theorem-6} the estimated value of radius of the ball $\Omega(0, r)$ is $r = 0.5265$. The output $y(k)$, the switching function $s(k)$ and control input $u(k)$ of the system with disturbance using nonlinear surface are shown in Fig.4, Fig.5 and Fig.6.

\section*{REFERENCES}


